***SCHAUM'S OUTLINE OF***

**THEORY AND PROBLEMS**

of

# THEORETICAL MECHANICS

**SI (METRIC) EDITION**

with an introduction to
Lagrange's Equations
and Hamiltonian Theory

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by

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$$F_1 = \frac{\partial\phi}{\partial x}, \quad F_2 = \frac{\partial\phi}{\partial y}, \quad F_3 = \frac{\partial\phi}{\partial z}$$

and so $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi$. Thus $\nabla\times\mathbf{F} = \nabla\times\nabla\phi = \mathbf{0}$.

Conversely if $\nabla\times\mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \nabla\phi$ and so $\mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r} = d\phi$, i.e. $F_1\,dx + F_2\,dy + F_3\,dz = d\phi$, an exact differential.

(b) $\mathbf{F} = (y^2z^3\cos x - 4x^3z)\,\mathbf{i} + 2z^3y\sin x\,\mathbf{j} + (3y^2z^2\sin x - x^4)\,\mathbf{k}$ and $\nabla\times\mathbf{F}$ is computed to be zero, so that by part (a) the required result follows.

**2.34.** Referring to Problem 2.4 find (a) the kinetic energy of the particle at $t=1$ and $t=2$, (b) the work done by the field in moving the particle from the point where $t=1$ to the point where $t=2$, (c) the momentum of the particle at $t=1$ and $t=2$ and (d) the impulse in moving the particle from $t=1$ to $t=2$.

(a) From part (a) of Problem 2.4,

$$\mathbf{v} = (4t^3+6)\mathbf{i} + (9t^2-8t+15)\mathbf{j} - (3t^2+8)\mathbf{k}$$

Then the velocities at $t=1$ and $t=2$ are

$$\mathbf{v}_1 = 10\mathbf{i} + 16\mathbf{j} - 11\mathbf{k}, \qquad \mathbf{v}_2 = 38\mathbf{i} + 35\mathbf{j} - 20\mathbf{k}$$

and the kinetic energies at $t=1$ and $t=2$ are

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(2)[(10)^2+(16)^2+(-11)^2] = 477, \qquad T_2 = \tfrac{1}{2}mv_2^2 = 3069$$

(b) Work done $= \displaystyle\int_{t=1}^{2} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{t=1}^{2} [24t^2\mathbf{i} + (36t-16)\mathbf{j} - 12t\mathbf{k}]\cdot[(4t^3+6)\mathbf{i} + (9t^2-8t+15)\mathbf{j} - (3t^2+8)\mathbf{k}]\,dt$$

$$= \int_{t=1}^{2} [(24t^2)(4t^3+6) + (36t-16)(9t^2-8t+15) + (12t)(3t^2+8)]\,dt = 2592$$

Note that by part (a) this is the same as the difference or change in kinetic energies $3069 - 477 = 2592$, illustrating Theorem 2.1, page 35, that Work done = change in kinetic energy.

(c) By part (a) the momentum at any time $t$ is

$$\mathbf{p} = m\mathbf{v} = 2\mathbf{v} = (8t^3+12)\mathbf{i} + (18t^2-16t+30)\mathbf{j} - (6t^2+16)\mathbf{k}$$

Then the momenta at $t=1$ and $t=2$ are

$$\mathbf{p}_1 = 20\mathbf{i} + 32\mathbf{j} - 22\mathbf{k}, \qquad \mathbf{p}_2 = 76\mathbf{i} + 70\mathbf{j} - 40\mathbf{k}$$

(d) Impulse $= \displaystyle\int_{t=1}^{2} \mathbf{F}\,dt$

$$= \int_{t=1}^{2} [24t^2\mathbf{i} + (36t-16)\mathbf{j} - 12t\mathbf{k}]\,dt = 56\mathbf{i} + 38\mathbf{j} - 18\mathbf{k}$$

Note that by part (b) this is the same as the difference or change in momentum, i.e. $\mathbf{p}_2 - \mathbf{p}_1 = (76\mathbf{i}+70\mathbf{j}-40\mathbf{k}) - (20\mathbf{i}+32\mathbf{j}-22\mathbf{k}) = 56\mathbf{i} + 38\mathbf{j} - 18\mathbf{k}$, illustrating Theorem 2.6, page 36, that Impulse = change in momentum.

**2.35.** A particle of mass $m$ moves along the $x$ axis under the influence of a conservative force field having potential $V(x)$. If the particle is located at positions $x_1$ and $x_2$ at respective times $t_1$ and $t_2$, prove that if $E$ is the total energy,

$$t_2 - t_1 = \sqrt{\frac{m}{2}}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$

**2.99.** A particle of mass 3 units moves in the $xy$ plane under the influence of a force field having potential $V = 6x^3 + 12y^3 + 36xy - 48x^2$. Investigate the motion of the particle if it is displaced slightly from its equilibrium position.

[*Hint.* Near $x = 0, y = 0$ the potential is very nearly $36xy - 48x^2$ since $6x^3$ and $12y^3$ are negligible.]

**2.100.** A particle of unit mass moves on the $x$ axis under the influence of a force field having potential $V = 6x(x-2)$. (*a*) Show that $x = 1$ is a position of stable equilibrium. (*b*) Prove that if the mass is displaced slightly from its position of equilibrium it will oscillate about it with period equal to $4\pi\sqrt{3}$.

[*Hint.* Let $x = 1 + u$ and neglect terms in $u$ of degree higher than one.]

**2.101.** A particle of mass $m$ moves in a force field $\mathbf{F} = -\kappa x\mathbf{i}$. (*a*) How much work is done in moving the particle from $x = x_1$ to $x = x_2$? (*b*) If a unit particle starts at $x = x_1$, with speed $v_1$, what is its speed on reaching $x = x_2$? *Ans.* (*a*) $\frac{1}{2}\kappa(x_1^2 - x_2^2)$, (*b*) $\sqrt{v_1^2 + (\kappa/m)(x_1^2 - x_2^2)}$

**2.102.** A particle of mass 2 moves in the $xy$ plane under the influence of a force field having potential $V = x^2 + y^2$. The particle starts at time $t = 0$ from rest at the point $(2, 1)$. (*a*) Set up the differential equations and conditions describing the motion. (*b*) Find the position at any time $t$. (*c*) Find the velocity at any time $t$.

**2.103.** Work Problem 2.102 if $V = 8xy$.

**2.104.** Does Theorem 2.7, page 36, hold relative to a non-inertial frame of reference or coordinate system? Prove your answer.

**2.105.** (*a*) Prove that if a particle moves in the $xy$ plane under the influence of a force field having potential $V = 12x(3y - 4x)$, then $x = 0$, $y = 0$ is a point of stable equilibrium. (*b*) Discuss the relationship of the result in (*a*) to Problem 2.37, page 53.

**2.106.** (*a*) Prove that a sufficient condition for the point $(a, b)$ to be a minimum point of the function $V(x, y)$ is that at $(a, b)$

$$\text{(i)}\quad \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0, \qquad \text{(ii)}\quad \Delta = \left(\frac{\partial^2 V}{\partial x^2}\right)\left(\frac{\partial^2 V}{\partial y^2}\right) - \left(\frac{\partial^2 V}{\partial x\,\partial y}\right)^2 > 0 \quad\text{and}\quad \frac{\partial^2 V}{\partial x^2} > 0$$

(*b*) Use (*a*) to investigate the points of stability of a particle moving in a force field having potential $V = x^3 + y^3 - 3x - 12y$. *Ans.* (*b*) The point $(1, 2)$ is a point of stability

**2.107.** Suppose that a particle of unit mass moves in the force field of Problem 2.106. Find its speed at any time.

**2.108.** A particle moves once around the circle $\mathbf{r} = a(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$ in a force field

$$\mathbf{F} = (x\mathbf{i} - y\mathbf{j})/(x^2 + y^2)$$

(*a*) Find the work done. (*b*) Is the force field conservative? (*c*) Do your answers to (*a*) and (*b*) contradict Theorem 2.4, page 35? Explain.

**2.109.** It is sometimes stated that classical or Newtonian mechanics makes the assumption that space and time are both absolute. Discuss what is meant by this statement.

**2.110.** The quantity $\mathbf{F}_{av} = \dfrac{\int_{t_1}^{t_2} \mathbf{F}\,dt}{t_2 - t_1}$ is called the *average force* acting on a particle from time $t_1$ to $t_2$. Does the result (*3*) of Problem 2.5, page 40, hold if $\mathbf{F}$ is replaced by $\mathbf{F}_{av}$? Explain.

$$\mathbf{W} = (\mathbf{W}\cdot\mathbf{r}_1)\mathbf{r}_1 + (\mathbf{W}\cdot\boldsymbol{\theta}_1)\boldsymbol{\theta}_1$$
$$= (-mg\mathbf{j}\cdot\mathbf{r}_1)\mathbf{r}_1 + (-mg\mathbf{j}\cdot\boldsymbol{\theta}_1)\boldsymbol{\theta}_1 = -mg\sin\theta\,\mathbf{r}_1 - mg\cos\theta\,\boldsymbol{\theta}_1$$

Also, $$\mathbf{N} = N\mathbf{r}_1$$

Using Newton's second law and the result of Problem 1.49, page 26, we have

$$\mathbf{F} = m\mathbf{a} = m[(\ddot{r} - r\dot{\theta}^2)\mathbf{r}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{\theta}_1]$$
$$= \mathbf{W} + \mathbf{N} = (N - mg\sin\theta)\mathbf{r}_1 - mg\cos\theta\,\boldsymbol{\theta}_1 \tag{1}$$

Thus $$m(\ddot{r} - r\dot{\theta}^2) = N - mg\sin\theta, \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\cos\theta \tag{2}$$

While the particle is on the circle (or sphere), we have $r = b$. Substituting this into (*2*),

$$-mb\dot{\theta}^2 = N - mg\sin\theta, \qquad b\ddot{\theta} = -g\cos\theta \tag{3}$$

Multiplying the second equation by $\dot{\theta}$, we see that it can be written

$$b\frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = -g\frac{d}{dt}(\sin\theta)$$

Integrating, $b\dot{\theta}^2/2 = -g\sin\theta + c_1$. Now when $\theta = \pi/2$, $\dot{\theta} = 0$ so that $c_1 = g$ and

$$b\dot{\theta}^2 = 2g(1 - \sin\theta) \tag{4}$$

Substituting (*4*) into the first equation of (*3*), we find

$$N = mg(3\sin\theta - 2) \tag{5}$$

Now as long as $N > 0$ the particle stays on the sphere; but when $N = 0$ the particle will be just about to leave the sphere. Thus the required angle is given by $3\sin\theta - 2 = 0$, i.e.,

$$\sin\theta = 2/3 \quad \text{or} \quad \theta = \sin^{-1} 2/3 \tag{6}$$

(*b*) Putting $\sin\theta = \frac{2}{3}$ into (*4*), we find

$$\dot{\theta}^2 = 2g/3b \tag{7}$$

Then if $v$ is the speed, we have $v = b\dot{\theta}$ so that (*7*) yields $v^2 = \frac{2}{3}bg$ or $v = \sqrt{\frac{2}{3}bg}$.

**Method 2.** By the conservation of energy, using the $x$ axis as reference level, we have

$$\text{P.E. at } A + \text{K.E. at } A = \text{P.E. at } P + \text{K.E. at } P$$
$$mgb + 0 = mgb\sin\theta + \tfrac{1}{2}mv^2$$

or $$v^2 = 2gb(1 - \sin\theta) \tag{8}$$

Using the result of Problem 1.35, page 20, together with Newton's second law, we have, since the radius of curvature is $b$,

$$\mathbf{F} = m\mathbf{a} = \left(\frac{v^2}{b}\mathbf{r}_1 - \frac{dv}{dt}\boldsymbol{\theta}_1\right) = \mathbf{W} + \mathbf{N}$$
$$= (N - mg\sin\theta)\mathbf{r}_1 - mg\cos\theta\,\boldsymbol{\theta}_1$$

Using only the $\mathbf{r}_1$ component, we have

$$v^2/b = N - mg\sin\theta \tag{9}$$

From (*8*) and (*9*) we find $N = mg(3\sin\theta - 2)$ which yields the required angle $\sin^{-1}(\frac{2}{3})$ as in Method 1. The speed is then found from (*8*).

**3.60.** How much force is needed to pull a train weighing 320 tonnes from rest to a speed of 25 km/h in 20 seconds if the coefficient of friction is 0.02 and (*a*) the track is horizontal, (*b*) the track is inclined at an angle of 10° with the horizontal and the train is going upward? [Use $\sin 10° = .1737$, $\cos 10° = .9848$.] *Ans.* (*a*) $1.74 \times 10^5$ N, (*b*) $7.18 \times 10^5$ N

**3.61.** Work Problem 3.60(*b*) if the train is going down the incline. *Ans.* $3.72 \times 10^5$ N up the incline

**3.62.** A train of mass $m$ is coasting down an inclined plane of angle $\alpha$ and coefficient of friction $\mu$ with constant speed $v_0$. Prove that the force needed to stop the train in a time $\tau$ is given by $mg(\sin\alpha - \mu\cos\alpha) + mv_0/\tau$.

## MISCELLANEOUS PROBLEMS

**3.63.** A stone is dropped down a well and the sound of the splash is heard after time $\tau$. Assuming the speed of sound is $c$, prove that the depth of the water level in the well is $(\sqrt{c^2 + 2gc\tau} - c)^2/2g$.

**3.64.** A projectile is launched downward from the top of an inclined plane of angle $\alpha$ in a direction making an angle $\gamma$ with the incline. Assuming that the projectile hits the incline, prove that (*a*) the range is given by $R = \dfrac{2v_0^2 \sin\gamma \cos(\gamma - \alpha)}{g\cos^2\alpha}$ and that (*b*) the maximum range down the incline is $R_{max} = \dfrac{v_0^2}{g(1 - \sin\alpha)}$.

**3.65.** A cannon is located on a hill which has the shape of an inclined plane of angle $\alpha$ with the horizontal. A projectile is fired from this cannon in a direction up the hill and making an angle $\beta$ with it. Prove that in order for the projectile to hit the hill horizontally we must have $\beta = \tan^{-1}\left(\dfrac{2\sin 2\alpha}{3 - \cos 2\alpha}\right)$.

**3.66.** Suppose that two projectiles are launched at angles $\alpha$ and $\beta$ with the horizontal from the same place at the same time in the same vertical plane and with the same initial speed. Prove that during the course of the motion, the line joining the projectiles makes a constant angle with the vertical given by $\frac{1}{2}(\alpha + \beta)$.

**3.67.** Is it possible to solve equation (*1*), page 33, by the method of separation of variables? Explain.

**3.68.** When launched at angle $\theta_1$ with the horizontal a projectile falls a distance $D_1$ short of its target, while at angle $\theta_2$ it falls a distance $D_2$ beyond the target. Find the angle at which the projectile should be launched so as to hit the target.

**3.69.** An object was thrown vertically downward. During the tenth second of travel it fell twice as far as during the fifth second. With what speed was it thrown? *Ans.* 4.9 m/s

**3.70.** A gun of muzzle speed $v_0$ is situated at height $h$ above a horizontal plane. Prove that the angle at which it must be fired so as to achieve the greatest range on the plane is given by $\theta = \frac{1}{2}\cos^{-1} gh/(v_0^2 + gh)$.

**3.71.** In Fig. 3-25, $AB$ is a smooth table and masses $m_1$ and $m_2$ are connected by a string over the smooth peg at $B$. Find (*a*) the acceleration of mass $m_2$ and (*b*) the tension in the string.

*Ans.* (*a*) $\dfrac{m_2 - m_1}{m_1 + m_2} g$, $m_2 > m_1$

(*b*) $\dfrac{m_1 m_2}{m_1 + m_2} g$

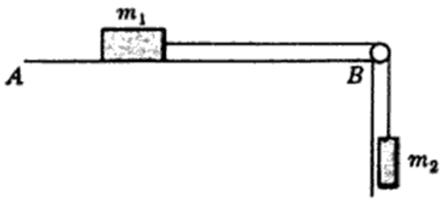


Fig. 3-25

**3.72.** Work Problem 3.71 if the table $AB$ has coefficient of friction $\mu$.

**3.73.** The maximum range of a projectile when fired down an inclined plane is twice the maximum range when fired up the inclined plane. Find the angle which the incline makes with the horizontal. *Ans.* $\sin^{-1} 1/3$

**4.103.** In Problem 4.96 suppose that $\omega_1/\omega_2$ is irrational and that at $t = 0$ the particle is at the particular point $(x_0, y_0)$ inside the rectangle defined by $x = \pm A$, $y = \pm B$. Prove that the point $(x_0, y_0)$ will never be reached again but that in the course of its motion the particle will come arbitrarily close to the point.

**4.104.** A particle oscillates on a vertical frictionless cycloid with its vertex downward. Prove that the projection of the particle on a vertical axis oscillates with simple harmonic motion.

**4.105.** A mass of 5 kg at the lower end of a vertical spring which has an elastic constant equal to 20 newtons/meter oscillates with a period of 10 seconds. Find (*a*) the damping constant, (*b*) the natural period and (*c*) the logarithmic decrement. *Ans.* (*a*) 19 N s/m, (*b*) 3.14 s

**4.106.** A mass of 100 g is supported in equilibrium by two identical springs of negligible mass having elastic constant equal to 50 dynes/cm. In the equilibrium position shown in Fig. 4-25 the springs make an angle of 30° with the horizontal and are 100 cm in length. If the mass is pulled down a distance of 2 cm and released, find the period of the resulting oscillation.

**4.107.** A thin hollow circular cylinder of inner radius 10 cm is fixed so that its axis is horizontal. A particle is placed on the inner frictionless surface of the cylinder so that its vertical distance above the lowest point of the inner surface is 2 cm. Find (*a*) the time for the particle to reach the lowest point and (*b*) the period of the oscillations which take place.

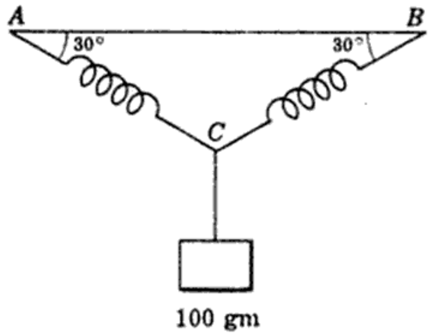


Fig. 4-25

**4.108.** A cubical box of side $a$ and weight $W$ vibrates vertically in water of density $\sigma$. Prove that the period of vibration is $(2\pi/a)\sqrt{\dfrac{W}{\sigma g^2}}$

**4.109.** A spring vibrates so that its equation of motion is

$$m\, d^2x/dt^2 + \kappa x = F(t)$$

If $x = 0$, $dx/dt = 0$ at $t = 0$, find $x$ as a function of time $t$.

*Ans.* $x = \dfrac{1}{\sqrt{m\kappa}} \displaystyle\int_0^t F(u) \sin \sqrt{\kappa/m}\,(t-u)\, du$

**4.110.** Work Problem 4.109 if damping proportional to $dx/dt$ is taken into account.

**4.111.** A spring vibrates so that its equation of motion is

$$m\, d^2x/dt^2 + \kappa x = 5 \cos \omega t + 2 \cos 3\omega t$$

If $x = 0$, $\dot{x} = v_0$ at $t = 0$, (*a*) find $x$ at any time $t$ and (*b*) determine for what values of $\omega$ resonance will occur.

**4.112.** A vertical spring having elastic constant $\kappa$ carries a mass $m$ at its lower end. At $t = 0$ the spring is in equilibrium and its upper end is suddenly made to move vertically so that its distance from the original point of support is given by $A \sin \omega t$, $t \geqq 0$. Find (*a*) the position of the mass $m$ at any time and (*b*) the values of $\omega$ for which resonance occurs.

**4.113.** (*a*) Solve $d^2x/dt^2 + x = t \sin t + \cos t$ where $x = 0$, $dx/dt = 0$ at $t = 0$, and (*b*) give a physical interpretation.

**4.114.** Discuss the motion of a simple pendulum for the case where damping and external forces are present.

for example,
$$V = -\int f(r)\,dr \qquad (2)$$
it follows that the field is conservative and that (2) represents the potential or potential energy.

Method 2.

We can show that $\nabla \times \mathbf{F} = 0$ directly, but this method is tedious although straightforward.

**5.9.** **Write the conservation of energy for a particle of mass $m$ in a central force field.**

Method 1. The velocity of a particle expressed in polar coordinates is [Problem 1.49, page 27]
$$\mathbf{v} = \dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1 \qquad \text{so that} \qquad v^2 = \mathbf{v}\cdot\mathbf{v} = \dot{r}^2 + r^2\dot{\theta}^2$$
Then the principle of conservation of energy can be expressed as
$$\tfrac{1}{2}mv^2 + V = E \qquad \text{or} \qquad \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E$$
where $E$ is a constant.

Method 2. The equations of motion for a particle in a central field are, by Problem 5.3,
$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad (1)$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (2)$$
Multiply equation (1) by $\dot{r}$, equation (2) by $r\dot{\theta}$ and add to obtain
$$m(\dot{r}\ddot{r} + r^2\dot{\theta}\ddot{\theta} + r\dot{r}\dot{\theta}^2) = f(r)\dot{r} \qquad (3)$$
This can be written
$$\tfrac{1}{2}m\frac{d}{dt}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{d}{dt}\int f(r)\,dr \qquad (4)$$
Then integrating both sides, we obtain
$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E \qquad (5)$$

**5.10.** **Show that the differential equation describing the motion of a particle in a central field can be written as**
$$\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \int f(r)\,dr = E$$

From Problem 5.9 we have by the conservation of energy,
$$\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \int f(r)\,dr = E \qquad (1)$$
We also have
$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} \qquad (2)$$
Substituting (2) into (1), we find
$$\tfrac{1}{2}m\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2 - \int f(r)\,dr = E \qquad \text{or} \qquad \frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right] - \int f(r)\,dr = E$$
since $\dot{\theta} = h/r^2$.

**5.11.** **(a) If $u = 1/r$, prove that $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\{(du/d\theta)^2 + u^2\}$.**

**(b) Use (a) to prove that the conservation of energy equation becomes**
$$(du/d\theta)^2 + u^2 = 2(E - V)/mh^2$$

(a) From equations (1) and (3) of Problem 5.7 we have $\dot{\theta} = hu^2$, $\dot{r} = -h\,du/d\theta$. Thus

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2(du/d\theta)^2 + (1/u^2)(hu^2)^2 = h^2\{(du/d\theta)^2 + u^2\}$$

(b) From the conservation of energy [Problem 5.9] and part (a),

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(\dot{r}^2 + r\dot{\theta}^2) = E - V \qquad \text{or} \qquad (du/d\theta)^2 + u^2 = 2(E-V)/mh^2$$

## DETERMINATION OF ORBIT FROM CENTRAL FORCE, OR CENTRAL FORCE FROM ORBIT

**5.12.** Show that the position of the particle as a function of time $t$ can be determined from the equations

$$t = \int [G(r)]^{-1/2}\,dr, \qquad t = \frac{1}{h}\int r^2\,d\theta$$

where

$$G(r) = \frac{2E}{m} + \frac{2}{m}\int f(r)\,dr - \frac{h^2}{2m^2r^2}$$

Placing $\dot{\theta} = h/r^2$ in the equation for conservation of energy of Problem 5.9,

$$\tfrac{1}{2}m(\dot{r}^2 + h^2/r^2) - \int f(r)\,dr = E$$

or

$$\dot{r}^2 = \frac{2E}{m} + \frac{2}{m}\int f(r)\,dr - \frac{h^2}{r^2} = G(r)$$

Then assuming the positive square root, we have

$$dr/dt = \sqrt{G(r)}$$

and so separating the variables and integrating, we find

$$t = \int [G(r)]^{-1/2}\,dr$$

The second equation follows by writing $\dot{\theta} = h/r^2$ as $dt = r^2\,d\theta/h$ and integrating.

**5.13.** Show that if the law of central force is defined by

$$f(r) = -K/r^2, \qquad K > 0$$

i.e. an inverse square law of attraction, then the path of the particle is a conic.

**Method 1.**

In this case $f(1/u) = -Ku^2$. Substituting into the differential equation of motion in Problem 5.7, we find

$$d^2u/d\theta^2 + u = K/mh^2 \tag{1}$$

This equation has the general solution

$$u = A\cos\theta + B\sin\theta + K/mh^2 \tag{2}$$

or using Problem 4.2, page 92,

$$u = K/mh^2 + C\cos(\theta - \phi) \tag{3}$$

i.e.,

$$r = \frac{1}{K/mh^2 + C\cos(\theta-\phi)} \tag{4}$$

It is always possible to choose the axes so that $\phi = 0$, in which case we have

$$r = \frac{1}{K/mh^2 + C\cos\theta} \tag{5}$$

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**5.87.** Find the force of attraction of an infinitely long thin uniform rod on a mass $m$ at distance $b$ from it. *Ans.* Magnitude is $2Gm\sigma/b$

**5.88.** A uniform wire is in the form of an arc of a circle of radius $b$ and central angle $\psi$. Prove that the force of attraction of the wire on a mass $m$ placed at the center of the circle is given in magnitude by

$$\frac{2GMm\sin(\psi/2)}{b^2\psi} \quad \text{or} \quad \frac{2G\sigma m\sin(\psi/2)}{b}$$

where $M$ is the mass of the wire and $\sigma$ is the mass per unit length. Discuss the cases $\psi = \pi/2$ and $\psi = \pi$.

**5.89.** In Fig. 5-18, $AB$ is a thin rod of length $2a$ and $m$ is a mass located at point $C$ a distance $b$ from the rod. Prove that the force of attraction of the rod on $m$ has magnitude

$$\frac{GMm}{ab}\sin\tfrac{1}{2}(\alpha+\beta)$$

in a direction making an angle with the rod given by

$$\tan^{-1}\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha-\cos\beta}\right)$$

Discuss the case $\alpha = \beta$ and compare with Problem 5.26.

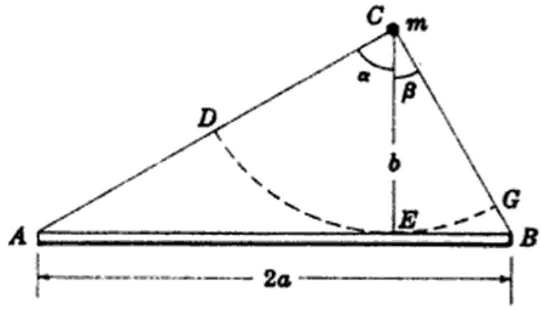


Fig. 5-18

**5.90.** By comparing Problem 5.89 with Problem 5.88, prove that the rod of Problem 5.89 can be replaced by a wire in the form of circular arc $DEG$ [shown dashed in Fig. 5-18] which has its center at $C$ and is tangent to the rod at $E$. Prove that the direction of the attraction is toward the midpoint of this arc.

**5.91.** A hemisphere of mass $M$ and radius $a$ has a particle of mass $m$ located at its center. Find the force of attraction if (*a*) the hemisphere is a thin shell, (*b*) the hemisphere is solid.
*Ans.* (*a*) $GMm/2a^2$, (*b*) $3GMm/2a^2$

**5.92.** Work Problem 5.91 if the hemisphere is a shell having outer radius $a$ and inner radius $b$.

**5.93.** Deduce from Kepler's laws that if the force of attraction between sun and planets is given in magnitude by $\gamma m/r^2$, then $\gamma$ must be independent of the particular planet.

**5.94.** A cone has height $H$ and radius $a$. Prove that the force of attraction on a particle of mass $m$ placed at its vertex has magnitude $\dfrac{6GMm}{a^2}\left(1-\dfrac{H}{\sqrt{a^2+H^2}}\right)$.

**5.95.** Find the force of attraction between two non-intersecting spheres.

**5.96.** A particle of mass $m$ is placed outside of a uniform solid hemisphere of radius $a$ at a distance $a$ on a line perpendicular to the base through its center. Prove that the force of attraction is given in magnitude by $GMm(\sqrt{2}-1)/a^2$.

**5.97.** Work (*a*) Problem 5.26, (*b*) Problem 5.27, and (*c*) Problem 5.94 by first finding the potential.

## MISCELLANEOUS PROBLEMS

**5.98.** A particle is projected vertically upward from the earth's surface with initial speed $v_0$.

(*a*) Prove that the maximum height $H$ reached above the earth's surface is $H = v_0^2R/(2gR - v_0^2)$.

(*b*) Discuss the significance of the case where $v_0^2 = 2gR$.

(*c*) Prove that if $H$ is small, then it is equal to $v_0^2/2g$ very nearly.

5.113. Work Problems 5.111 and 5.112 if the hole is straight but does not pass through the center of the earth.

5.114. Discuss the relationship between the results of Problems 5.111 and 5.112 and that of Problem 5.39.

5.115. How would you explain the fact that the earth has an atmosphere while the moon has none?

5.116. Prove Theorem 5.1, page 120.

5.117. Discuss Theorem 5.1 if the spheres intersect.

5.118. Explain how you could use the result of Problem 5.27 to find the force of attraction of a solid sphere on a particle.

5.119. Find the force of attraction between a uniform circular ring of outer radius $a$ and inner radius $b$ and a mass $m$ located on its axis at a distance $b$ from its center.

5.120. Two space ships move about the earth on the same elliptical path of eccentricity $\epsilon$. If they are separated by a small distance $D$ at perigee, prove that at apogee they will be separated by the distance $D(1-\epsilon)/(1+\epsilon)$.

5.121. (*a*) Explain how you could calculate the velocity of escape from a planet. (*b*) Use your method to calculate the velocity of escape from Mars. *Ans.* (*b*) 5 km/s

5.122. Work Problem 5.121 for (*a*) Jupiter, (*b*) Venus. *Ans.* (*a*) about 61 km/s, (*b*) about 10 km/s

5.123. Three infinitely long thin uniform rods having the same mass per unit length lie in the same plane and form a triangle. Prove that force of attraction on a particle will be zero if and only if the particle is located at the intersection of the medians of the triangle.

5.124. Find the force of attraction between a uniform rod of length $a$ and a sphere of radius $b$ if they do not intersect and the line of the rod passes through the center.

5.125. Work Problem 5.124 if the rod is situated so that a line drawn from the center perpendicular to the line of the rod bisects the rod.

5.126. A satellite of radius $a$ revolves in a circular orbit about a planet of radius $b$ with period $P$. If the shortest distance between their surfaces is $c$, prove that the mass of the planet is $4\pi^2(a+b+c)^3/GP^2$.

5.127. Given that the moon is approximately 386,000 km from the earth and makes one complete revolution about the earth in $27\frac{1}{3}$ days approximately, find the mass of the earth.
*Ans.* $6 \times 10^{24}$ kg

5.128. Discuss the relationship of Problem 5.126 with Kepler's third law.

5.129. Prove that the only central force field $\mathbf{F}$ whose divergence is zero is an inverse square force field.

5.130. Work Problem 5.32, page 132, by using triple integration.

5.131. A uniform solid right circular cylinder has radius $a$ and height $H$. A particle of mass $m$ is placed on the extended axis of the cylinder so that it is at a distance $D$ from one end. Prove that the force of attraction is directed along the axis and given in magnitude by

$$\frac{2GMm}{a^2H}\{H + \sqrt{a^2+D^2} - \sqrt{a^2+(D+H)^2}\}$$

5.132. Suppose that the cylinder of Problem 5.131 has a given volume. Prove that the force of attraction when the particle is at the center of one end of the cylinder is a maximum when $a/H = \frac{1}{8}(9-\sqrt{17})$.

5.133. Work (*a*) Problem 5.26 and (*b*) Problem 5.27 assuming an inverse cube law of attraction.

## VELOCITY IN A MOVING SYSTEM

If, in particular, vector **A** is the position vector **r** of a particle, then (*2*) gives

$$\left.\frac{d\mathbf{r}}{dt}\right|_F = \left.\frac{d\mathbf{r}}{dt}\right|_M + \boldsymbol{\omega}\times\mathbf{r} \tag{4}$$

or

$$D_F\mathbf{r} = D_M\mathbf{r} + \boldsymbol{\omega}\times\mathbf{r} \tag{5}$$

Let us write

$\mathbf{v}_{P|F} = d\mathbf{r}/dt\,|_F = D_F\mathbf{r}$ = velocity of particle $P$ relative to fixed system

$\mathbf{v}_{P|M} = d\mathbf{r}/dt\,|_M = D_M\mathbf{r}$ = velocity of particle $P$ relative to moving system

$\mathbf{v}_{M|F} = \boldsymbol{\omega}\times\mathbf{r}$ = velocity of moving system relative to fixed system.

Then (*4*) or (*5*) can be written

$$\mathbf{v}_{P|F} = \mathbf{v}_{P|M} + \mathbf{v}_{M|F} \tag{6}$$

## ACCELERATION IN A MOVING SYSTEM

If $D_F^2 = d^2/dt^2\,|_F$ and $D_M^2 = d^2/dt^2\,|_M$ are second derivative operators with respect to $t$ in the fixed and moving systems, then application of (*3*) yields [see Problem 6.6]

$$D_F^2\mathbf{r} = D_M^2\mathbf{r} + (D_M\boldsymbol{\omega})\times\mathbf{r} + 2\boldsymbol{\omega}\times D_M\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{7}$$

Let us write

$\mathbf{a}_{P|F} = d^2\mathbf{r}/dt^2\,|_F = D_F^2\mathbf{r}$ = acceleration of particle $P$ relative to fixed system

$\mathbf{a}_{P|M} = d^2\mathbf{r}/dt^2\,|_M = D_M^2\mathbf{r}$ = acceleration of particle $P$ relative to moving system

$\mathbf{a}_{M|F} = (D_M\boldsymbol{\omega})\times\mathbf{r} + 2\boldsymbol{\omega}\times D_M\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$
= acceleration of moving system relative to fixed system

Then (*7*) can be written

$$\mathbf{a}_{P|F} = \mathbf{a}_{P|M} + \mathbf{a}_{M|F} \tag{8}$$

## CORIOLIS AND CENTRIPETAL ACCELERATION

The last two terms on the right of (*7*) are called the *Coriolis acceleration* and *centripetal acceleration* respectively, i.e.,

$$\text{Coriolis acceleration} = 2\boldsymbol{\omega}\times D_M\mathbf{r} = 2\boldsymbol{\omega}\times\mathbf{v}_M \tag{9}$$

$$\text{Centripetal acceleration} = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{10}$$

The second term on the right of (*7*) is sometimes called the *linear acceleration*, i.e.,

$$\text{Linear acceleration} = (D_M\boldsymbol{\omega})\times\mathbf{r} = \left(\left.\frac{d\boldsymbol{\omega}}{dt}\right|_M\right)\times\mathbf{r} \tag{11}$$

and $D_M\boldsymbol{\omega}$ is called the *angular acceleration*. For many cases of practical importance [e.g. in the rotation of the earth] $\boldsymbol{\omega}$ is constant and $D_M\boldsymbol{\omega} = 0$.

The quantity $-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is often called the *centrifugal acceleration.*

## MOTION OF A PARTICLE RELATIVE TO THE EARTH

Newton's second law is strictly applicable only to inertial systems. However, by using (*7*) we obtain a result valid for non-inertial systems. This has the form

$$mD_M^2\mathbf{r} = \mathbf{F} - m(D_M\boldsymbol{\omega})\times\mathbf{r} - 2m(\boldsymbol{\omega}\times D_M\mathbf{r}) - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) \tag{12}$$

where **F** is the resultant of all forces acting on the particle as seen by the observer in the fixed or inertial system.

**7.40.** Three equal masses are located at the vertices of a triangle. Prove that the center of mass is located at the intersection of the medians of the triangle.

**7.41.** A uniform plate has the shape of the region bounded by the parabola $y = x^2$ and the line $y = H$ in the $xy$ plane. Find the center of mass.
*Ans.* $\bar{x} = 0$, $\bar{y} = \frac{3}{5}H$

**7.42.** Find the center of mass of a uniform right circular cone of radius $a$ and height $H$.
*Ans.* That point on the axis at distance $\frac{3}{4}H$ from the vertex.

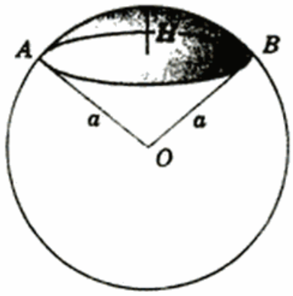


Fig. 7-19

**7.43.** The shaded region of Fig. 7-19 is a solid spherical cap of height $H$ cut off from a uniform solid sphere of radius $a$. (a) Prove that the centroid of the cap is located at a distance $(4aH - H^2)/(12a - 4H)$ from the base $AB$. (b) Discuss the cases $H = 0$, $H = a$ and $H = 2a$.

**7.44.** Find the center of mass of a uniform plate bounded by $y = \sin x$ and the $x$ axis. *Ans.* $\bar{x} = \pi/2$, $\bar{y} = \pi/8$

**7.45.** Find the center of mass of a rod of length $l$ whose density is proportional to the distance from one end $O$.
*Ans.* $\frac{2}{3}l$ from end $O$

**7.46.** Find the centroid of a uniform solid bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$.
*Ans.* $\bar{\mathbf{r}} = \frac{1}{16}a(\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$

**7.47.** A uniform solid is bounded by the paraboloid of revolution $x^2 + y^2 = cz$ and the plane $z = H$ [see Fig. 7-20]. Find the centroid. *Ans.* $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = \frac{2}{3}H$

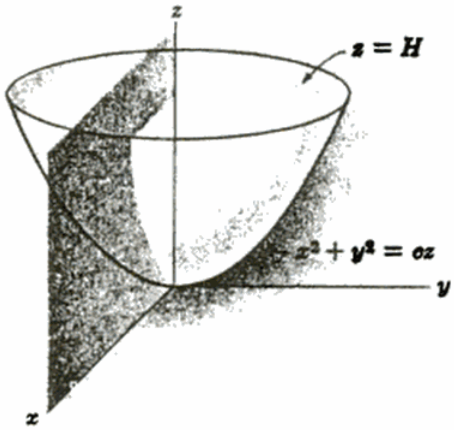


Fig. 7-20

## ANGULAR MOMENTUM AND TORQUE

**7.48.** Three particles of masses 2, 3 and 5 move under the influence of a force field so that their position vectors relative to a fixed coordinate system are given respectively by $\mathbf{r}_1 = 2t\mathbf{i} - 3\mathbf{j} + t^2\mathbf{k}$, $\mathbf{r}_2 = (t+1)\mathbf{i} + 3t\mathbf{j} - 4\mathbf{k}$ and $\mathbf{r}_3 = t^2\mathbf{i} - t\mathbf{j} + (2t-1)\mathbf{k}$ where $t$ is the time. Find (a) the total angular momentum of the system and (b) the total external torque applied to the system, taken with respect to the origin.
*Ans.* (a) $(31 - 12t)\mathbf{i} + (6t^2 - 10t - 12)\mathbf{j} + (21 + 5t^2)\mathbf{k}$
(b) $-12\mathbf{i} + (12t - 10)\mathbf{j} + 10t\mathbf{k}$

**7.49.** Work Problem 7.48 if the total angular momentum and torque are taken with respect to the center of mass.

**7.50.** Verify that in (a) Problem 7.48 and (b) Problem 7.49 the total external torque is equal to the time rate of change in angular momentum.

**7.51.** In Problem 7.48 find (a) the total angular momentum and (b) the total external torque taken about a point whose position vector is given by $\mathbf{r} = t\mathbf{i} - 2t\mathbf{j} + 3\mathbf{k}$. Does the total external torque equal the time rate of change in angular momentum in this case? Explain.

**7.52.** Verify Theorem 7.9, page 169, for the system of particles of Problem 7.48.

**7.53.** State and prove a theorem analogous to that of Theorem 7.9, page 169, for the total external torque applied to a system.

**7.54.** Is the angular momentum conserved in Problem 7.38? Explain.

The quantity $\epsilon$, called the *coefficient of restitution*, depends on the materials of which the objects are made and is generally taken as a constant between 0 and 1. If $\epsilon = 0$ the collision is called *perfectly inelastic* or briefly *inelastic*. If $\epsilon = 1$ the collision is called *perfectly elastic* or briefly *elastic*.

In the case of perfectly elastic collisions the total kinetic energy before and after impact is the same.

## CONTINUOUS SYSTEMS OF PARTICLES

For some problems the number of particles per unit length, area or volume is so large that for all practical purposes the system can be considered as continuous. Examples are a vibrating violin string, a vibrating drumhead or membrane, or a sphere rolling down an inclined plane.

The basic laws of Chapter 7 hold for such continuous systems of particles. In applying them, however, it is necessary to use integration in place of summation over the number of particles and the concept of density.

## THE VIBRATING STRING

Let us consider an elastic string such as a violin or piano string which is tightly stretched between the fixed points $x = 0$ and $x = l$ of the $x$ axis [see Fig. 8-1]. If the string is given some initial displacement [such as, for example, by plucking it] and is then released, it will vibrate or oscillate about the equilibrium position.

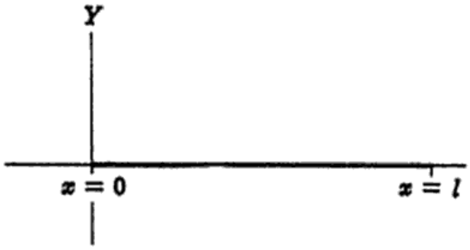


Fig. 8-1

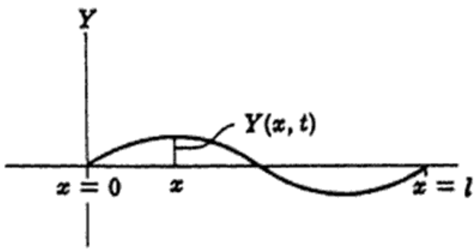


Fig. 8-2

If we let $Y(x, t)$ denote the displacement of any point $x$ of the string from the equilibrium position at time $t$ [see Fig. 8-2], then the equation governing the vibrations is given by the *partial differential equation*

$$\frac{\partial^2 Y}{\partial t^2} = c^2 \frac{\partial^2 Y}{\partial x^2} \tag{1}$$

where if $T$ is the (constant) tension throughout the string and $\sigma$ is the (constant) density [mass per unit length of string],

$$c^2 = T/\sigma \tag{2}$$

The equation (*1*) holds in case the vibrations are assumed so small that the slope $\partial Y/\partial x$ at any point of the string is much less than one.

## BOUNDARY-VALUE PROBLEMS

The problem of solving an equation such as (*1*) subject to various conditions, called boundary conditions, is often called a *boundary-value problem*. An important method for solving such problems makes use of *Fourier series*.

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## IMPULSE. CONSERVATION OF ANGULAR MOMENTUM

The time integral of the torque

$$\mathcal{J} = \int_{t_1}^{t_2} \mathbf{\Lambda}\, dt \tag{15}$$

is called the *angular impulse* from time $t_1$ to $t_2$.

We have the following theorems.

***Theorem 9.9.*** The angular impulse is equal to the change in angular momentum. In symbols

$$\int_{t_1}^{t_2} \mathbf{\Lambda}\, dt = \mathbf{\Omega}_2 - \mathbf{\Omega}_1 \tag{16}$$

***Theorem 9.10*: *Conservation of Angular Momentum.*** If the net torque applied to a rigid body is zero, then the angular momentum is constant, i.e. is conserved.

## THE COMPOUND PENDULUM

Let $\mathcal{R}$ [Fig. 9-5] be a rigid body which is free to oscillate in a vertical plane about a fixed horizontal axis through $O$ under the influence of gravity. We call such a rigid body a *compound pendulum.*

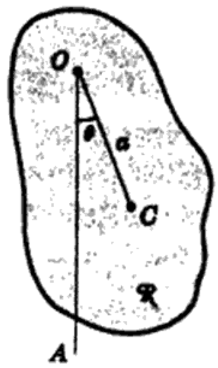


Fig. 9-5

Let $C$ be the center of mass and suppose that the angle between $OC$ and the vertical $OA$ is $\theta$. Then if $I_0$ is the moment of inertia of $\mathcal{R}$ about the horizontal axis through $O$, $M$ is the mass of the rigid body and $a$ is the distance $OC$, we have for the equation of motion,

$$\ddot{\theta} + \frac{Mga}{I_0}\sin\theta = 0 \tag{17}$$

For small oscillations the period of vibration is

$$P = 2\pi\sqrt{I_0/Mga} \tag{18}$$

The length of the equivalent simple pendulum is

$$l = I_0/Ma \tag{19}$$

The following theorem is of interest.

***Theorem 9.11.*** The period of vibration of a compound pendulum is a minimum when the distance $OC = a$ is equal to the radius of gyration of the body about the horizontal axis through the center of mass.

## GENERAL PLANE MOTION OF A RIGID BODY

The general plane motion of a rigid body can be considered as a translation parallel to the plane plus a rotation about a suitable axis perpendicular to the plane. Two important methods for treating general plane motion of a rigid body are given by the following theorems.

***Theorem 9.12*: *Principle of Linear Momentum.*** If $\mathbf{r}$ is the position vector of the center of mass of a rigid body relative to an origin $O$, then

$$\frac{d}{dt}(M\dot{\mathbf{r}}) = M\ddot{\mathbf{r}} = \mathbf{F} \tag{20}$$

where $M$ is the total mass, assumed constant, and $\mathbf{F}$ is the net external force acting on the body.

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where $E$ is constant. The result is equivalent to the principle of conservation of energy, since the kinetic energy is

$$T = \tfrac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \tfrac{1}{2}I_3A^2 \qquad (2)$$

while the potential energy is $$V = mgl\cos\theta \qquad (3)$$

and $T + V = E$ is the total energy.

(b) Multiplying the result of Problem 10.27(b) by $\sin\theta$,

$$I_1\ddot{\phi}\sin^2\theta + 2I_1\dot{\phi}\dot{\theta}\sin\theta\cos\theta - I_3A\dot{\theta}\sin\theta = 0$$

which can be written

$$\frac{d}{dt}(I_1\dot{\phi}\sin^2\theta + I_3A\cos\theta) = 0$$

Integrating, $$I_1\dot{\phi}\sin^2\theta + I_3A\cos\theta = \text{constant} = K \qquad (4)$$

To interpret this result physically, we note that the vertical component of the angular momentum is $I_1\dot{\phi}\sin^2\theta + I_3A\cos\theta$ [see Problem 10.123], and this must be constant since the torque due to the weight of the top has zero component in the vertical direction.

**10.30.** Let $u = \cos\theta$. Prove that:

(a) $$\dot{u}^2 = (\alpha - \beta u)(1 - u^2) - (\gamma - \delta u)^2 = f(u)$$

where $\alpha = 2(E - \tfrac{1}{2}I_3A^2)/I_1$, $\beta = 2mgl/I_1$, $\gamma = K/I_1$, $\delta = I_3A/I_1$;

(b) $$t = \int \frac{du}{\sqrt{f(u)}} + \text{constant}$$

(a) From Problem 10.29,

$$\tfrac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \tfrac{1}{2}I_3A^2 + mgl\cos\theta = E \qquad (1)$$

$$I_1\dot{\phi}\sin^2\theta + I_3A\cos\theta = K \qquad (2)$$

From (2), $$\dot{\phi} = \frac{K - I_3A\cos\theta}{I_1\sin^2\theta} \qquad (3)$$

Substituting this into (1),

$$\tfrac{1}{2}I_1\dot{\theta}^2 + \frac{(K - I_3A\cos\theta)^2}{2I_1\sin^2\theta} + \tfrac{1}{2}I_3A^2 + mgl\cos\theta = E$$

Letting $u = \cos\theta$ so that $\dot{u} = -\sin\theta\,\dot{\theta}$ and $\sin^2\theta = 1 - u^2$, this becomes

$$\tfrac{1}{2}I_1\frac{\dot{u}^2}{1-u^2} + \frac{(K - I_3Au)^2}{2I_1(1-u^2)} + mglu = E - \tfrac{1}{2}I_3A^2$$

Thus $$\dot{u}^2 + \left(\frac{K - I_3Au}{I_1}\right)^2 + \frac{2mglu(1-u^2)}{I_1} = \frac{2(1-u^2)}{I_1}(E - \tfrac{1}{2}I_3A^2)$$

which can be written as

$$\dot{u}^2 = (\alpha - \beta u)(1 - u^2) - (\gamma - \delta u)^2 = f(u) \qquad (4)$$

where $$\alpha = (2E - I_3A^2)/I_1, \quad \beta = 2mgl/I_1, \quad \gamma = K/I_1, \quad \delta = I_3A/I_1 \qquad (5)$$

Note that with this notation (3) can be written

$$\dot{\phi} = \frac{\gamma - \delta u}{1 - u^2} \qquad (6)$$

(b) From the result of (a) we have, since $\dot{u} > 0$,

$$\dot{u} = \frac{du}{dt} = \sqrt{f(u)} \quad \text{or} \quad dt = \frac{du}{\sqrt{f(u)}}$$

Integrating,
$$t = \int \frac{du}{\sqrt{f(u)}} + c \tag{7}$$

The integral can be evaluated in terms of *elliptic functions* which are periodic.

**10.31.** (a) Prove that $\dot{\theta} = 0$ at those values of $u$ for which
$$f(u) = (\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2 = 0$$

(b) Prove that the equation in (a) has three real roots $u_1, u_2, u_3$ but that in general not all the angles corresponding to these are real.

(a) From Problem 10.30(a), $\quad \dot{u}^2 = f(u) = (\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2 \qquad (1)$

Since $\dot{u} = -\sin\theta\,\dot{\theta}$, it follows that $\dot{\theta} = 0$ where $\dot{u} = 0$ or $f(u) = 0$. Thus $\dot{\theta} = 0$ at the roots of the equation
$$f(u) = (\alpha - \beta u)(1-u^2) - (\gamma - \delta u)^2 = 0 \tag{2}$$

(b) Equation (1) can be written as
$$f(u) = \beta u^3 - (\delta^2 + \alpha)u^2 + (2\gamma\delta - \beta)u + \alpha - \gamma^2 \tag{3}$$

Since $\beta > 0$, it follows that
$$f(\infty) = \infty, \quad f(-\infty) = -\infty$$
$$f(1) = -(\gamma - \delta)^2, \quad f(-1) = -(\gamma + \delta)^2$$

Thus there is a change of sign from $-$ to $+$ as $u$ goes from 1 to $\infty$, and consequently there must be a root, say $u_3$, between 1 and $\infty$ as indicated in Fig. 10-19.

Now we know that in order for the motion of the top to take place we must have $f(u) = \dot{u}^2 \geqq 0$. Also, since $0 \leqq \theta \leqq \pi/2$, we must have $0 \leqq u \leqq 1$. It thus follows that there must be two roots $u_1$ and $u_2$ between 0 and 1, as indicated in the figure.

It follows that in general there are two corresponding angles $\theta_1$ and $\theta_2$ such that $\cos\theta_1 = u_1$, $\cos\theta_2 = u_2$. In special cases it could happen that $u_1 = u_2$ or $u_2 = u_3 = 1$.

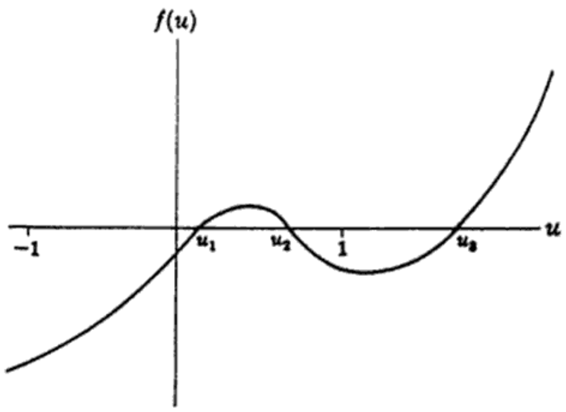


Fig. 10-19

**10.32.** Give a physical interpretation of the results found in Problem 10.31.

The fact that there are two roots $u_1$ and $u_2$ corresponding to $\theta_1$ and $\theta_2$ respectively, shows that the motion of the top is such that its axis always makes an angle $\theta$ with the vertical which lies between $\theta_1$ and $\theta_2$. This motion, which is a bobbing up and down of the axis between the limits $\theta_1$ and $\theta_2$, is called *nutation* and takes place at the same time as the *precessional motion* of the axis of the top about the vertical and the *spinning* of the top about its axis. Because the motion can be expressed in terms of elliptic functions [see Problem 10.104], we can show that it is periodic.

In general the tip of the axis of the top will describe one of various types of curves such as indicated in Figs. 10-20, 10-21 and 10-22. The type of curve will depend on the root of the equation [see equation (6) of Problem 10.30]
$$\dot{\phi} = \frac{\gamma - \delta u}{1-u^2} = 0 \tag{1}$$

**11.27.** Suppose that in Problem 11.26 the bead starts from rest at $A$. How long will it take to reach the end $B$ of the wire assuming that the length of the wire is $l$?

Since the bead starts from rest at $t = 0$, we have $r = 0$, $\dot{r} = 0$ at $t = 0$. Then from equation (2) of Problem 11.26,

$$c_1 + c_2 = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \quad \text{and} \quad c_1 - c_2 = 0$$

Thus $c_1 = c_2 = \dfrac{g \cos \alpha}{2\omega^2 \sin^2 \alpha}$ and (2) of Problem 11.26 becomes

$$r = \frac{g \cos \alpha}{2\omega^2 \sin^2 \alpha}\{e^{(\omega \sin \alpha)t} + e^{-(\omega \sin \alpha)t}\} - \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha} \tag{1}$$

or

$$r = \frac{g \cos \alpha}{\omega^2 \sin^2 \alpha}\{\cosh (\omega \sin \alpha)t - 1\} \tag{2}$$

which can also be obtained from equation (3) of Problem 11.26. When $r = l$, (2) yields

$$\cosh (\omega \sin \alpha)t = 1 + \frac{l\omega^2 \sin^2 \alpha}{g \cos \alpha}$$

so that the required time is

$$t = \frac{1}{\omega \sin \alpha} \cosh^{-1}\left(1 + \frac{l\omega^2 \sin^2 \alpha}{g \cos^2 \alpha}\right)$$

$$= \frac{1}{\omega \sin \alpha} \ln\left\{\left(1 + \frac{l\omega^2 \sin^2 \alpha}{g \cos^2 \alpha}\right) + \sqrt{\left(1 + \frac{l\omega^2 \sin^2 \alpha}{g \cos^2 \alpha}\right)^2 - 1}\right\}$$

**11.28.** A double pendulum [see Problem 11.1(*c*) and Fig. 11-3, page 285] vibrates in a vertical plane. (*a*) Write the Lagrangian of the system. (*b*) Obtain equations for the motion.

(*a*) The transformation equations given in Problem 11.2, page 286,

$$x_1 = l_1 \cos \theta_1 \qquad y_1 = l_1 \sin \theta_1$$

$$x_2 = l_1 \cos \theta_1 + l_2 \cos \theta_2 \qquad y_2 = l_1 \sin \theta_1 + l_2 \sin \theta_2$$

yield

$$\dot{x}_1 = -l_1\dot{\theta}_1 \sin \theta_1 \qquad \dot{y}_1 = l_1\dot{\theta}_1 \cos \theta_1$$

$$\dot{x}_2 = -l_1\dot{\theta}_1 \sin \theta_1 - l_2\dot{\theta}_2 \sin \theta_2 \qquad \dot{y}_2 = l_1\dot{\theta}_1 \cos \theta_1 + l_2\dot{\theta}_2 \cos \theta_2$$

The kinetic energy of the system is

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos (\theta_1 - \theta_2)]$$

The potential energy of the system [taking as reference level a plane at distance $l_1 + l_2$ below the point of suspension of Fig. 11-3] is

$$V = m_1g[l_1 + l_2 - l_1 \cos \theta_1] + m_2g[l_1 + l_2 - (l_1 \cos \theta_1 + l_2 \cos \theta_2)]$$

Then the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \tfrac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos (\theta_1 - \theta_2)] \\ &\quad - m_1g[l_1 + l_2 - l_1 \cos \theta_1] - m_2g[l_1 + l_2 - (l_1 \cos \theta_1 + l_2 \cos \theta_2)] \end{aligned} \tag{1}$$

(*b*) The Lagrange equations associated with $\theta_1$ and $\theta_2$ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{2}$$

**12.31.** Using Hamilton's equations, work the problem of the harmonic oscillator in (*a*) one dimension, (*b*) two dimensions, (*c*) three dimensions.

**12.32.** Work Problem 3.27, page 78 by using Hamilton's equation.

### PHASE SPACE AND LIOUVILLE'S THEOREM

**12.33.** Explain why the path of a phase point in phase space which represents the motion of a system of particles can never cross itself.

**12.34.** Carry out the details in the proof of Liouville's theorem for the case of two degrees of freedom.

### CALCULUS OF VARIATIONS AND HAMILTON'S PRINCIPLE

**12.35.** Use the methods of the calculus of variations to find that curve connecting two fixed points in a plane which has the shortest length.

**12.36.** Prove that if the function $F$ in the integral $\int_a^b F(x, y, y')\,dx$ is independent of $x$, then the integral is an extremum if $F - y'F_{y'} = c$ where $c$ is a constant.

**12.37.** Use the result of Problem 12.36 to solve (*a*) Problem 12.9, page 322, (*b*) Problem 12.35.

**12.38.** It is desired to revolve the curve of Fig. 12-5 having endpoints fixed at $P(x_1, y_1)$ and $Q(x_2, y_2)$ about the $x$ axis so that the area $I$ of the surface of revolution is a minimum.

(*a*) Show that $I = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$.

(*b*) Obtain the differential equation of the curve.

(*c*) Prove that the required curve is a catenary.

*Ans.* (*b*) $yy'' = 1 + (y')^2$

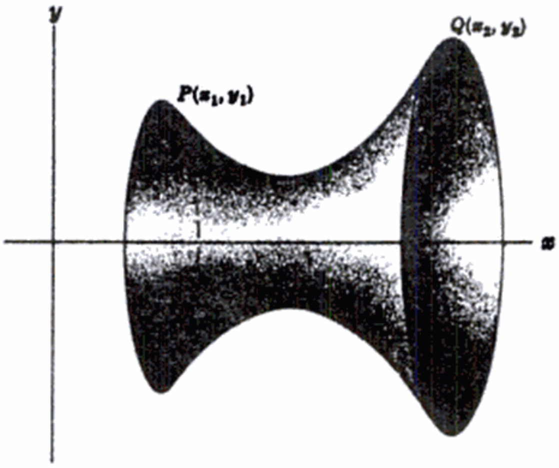


Fig. 12-5

**12.39.** Two identical circular wires in contact are placed in a soap solution and then separated so as to form a soap film. Explain why the shape of the soap film surface is related to the result of Problem 12.38.

**12.40.** Use Hamilton's principle to find the motion of a simple pendulum.

**12.41.** Work the problem of a projectile by using Hamilton's principle.

**12.42.** Use Hamilton's principle to find the motion of a solid cylinder rolling down an inclined plane of angle $\alpha$.

### CANONICAL TRANSFORMATIONS AND GENERATING FUNCTIONS

**12.43.** Prove that the transformation $Q = p,\ P = -q$ is canonical.

**12.44.** Prove that the transformation $Q = q\tan p,\ P = \ln\sin p$ is canonical.

**12.45.** (*a*) Prove that the Hamiltonian for a harmonic oscillator can be written in the form $H = \frac{1}{2}p^2/m + \frac{1}{2}\kappa q^2$.

(*b*) Prove that the transformation $q = \sqrt{2P/\sqrt{\kappa m}}\sin Q,\ p = \sqrt{2P\sqrt{\kappa m}}\cos Q$ is canonical.

(*c*) Express the Hamiltonian of part (*a*) in terms of $P$ and $Q$ and show that $Q$ is cyclic.

(*d*) Obtain the solution of the harmonic oscillator by using the above results.

## UNITS AND DIMENSIONS

| | Physical Quantity | Dimension | Name and Symbol of the SI Unit | Definition of the SI Unit | CGS System |
|---|---|---|---|---|---|
| Basic Units | Length | $L$ | meter (m) | m | cm |
| | Mass | $M$ | kilogram (kg) | kg | g |
| | Time | $T$ | second (s) | s | s |
| Derived Units | Velocity | $LT^{-1}$ | meter per second | m/s | cm/s |
| | Acceleration | $LT^{-2}$ | meter per second squared | m/s$^2$ | cm/s$^2$ |
| | Force | $MLT^{-2}$ | newton (N) | kg m s$^{-2}$ = J m$^{-1}$ | g cm s$^{-2}$ = dyn |
| | Momentum/Impulse | $MLT^{-1}$ | newton second | kg m s$^{-1}$ = N s | g cm s$^{-1}$ = dyn s |
| | Energy, Work | $ML^2T^{-2}$ | joule (J) | kg m$^2$s$^{-2}$ = N m | g cm$^2$s$^{-2}$ = erg = dyn cm |
| | Power | $ML^2T^{-3}$ | watt (W) | kg m$^2$s$^{-3}$ = J s$^{-1}$ | g cm$^2$s$^{-3}$ = erg s$^{-1}$ |
| | Volume | $L^3$ | cubic meter | m$^3$ | cm$^3$ |
| | Density | $ML^{-3}$ | kilogram per cubic meter | kg m$^{-3}$ | g cm$^{-3}$ |
| | Angle | — | radian (rad) | rad | rad |
| | Angular velocity | $T^{-1}$ | radian per second | rad s$^{-1}$ | rad s$^{-1}$ |
| | Angular acceleration | $T^{-2}$ | radian per second squared | rad s$^{-2}$ | rad s$^{-2}$ |
| | Torque or momentum of force | $ML^2T^{-2}$ | newton meter | kg m$^2$s$^{-2}$ = N m | g cm$^2$s$^{-2}$ = erg cm |
| | Angular momentum | $ML^2T^{-1}$ | newton meter per second | kg m$^2$s$^{-1}$ | g cm$^2$s$^{-1}$ |
| | Momentum of inertia | $ML^2$ | Kilogram meter squared | kg m$^2$ | g cm$^2$ |
| | Pressure | $ML^{-1}T^{-2}$ | pascal (Pa) | kg m$^{-1}$s$^{-2}$ = N m$^{-2}$ = J m$^{-3}$ | g cm$^{-1}$s$^{-2}$ = erg cm$^{-3}$ = dyn cm$^{-2}$ |
| | Frequency | $T^{-1}$ | hertz (Hz) | s$^{-1}$ | s$^{-1}$ |

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Particular solutions are often found from certain conditions imposed on the problem and sometimes called *boundary* or *initial conditions*. In Example 3 for instance, if we wish to satisfy the conditions $y = 5$ when $x = 0$ and $y' = dy/dx = 1$ when $x = 0$, we obtain $c_1 = 5$, $c_2 = -3$.

A problem in which we are required to solve a differential equation subject to given conditions is often called a *boundary-value problem*.

## SOLUTIONS TO SOME SPECIAL FIRST ORDER EQUATIONS

The following list shows some important methods for finding general solutions of first order differential equations.

**1. Separation of Variables**

If a first order equation can be written as

$$F(x)\,dx + G(y)\,dy = 0 \tag{1}$$

then the variables are said to be *separable* and the general solution obtained by direct integration is

$$\int F(x)\,dx + \int G(y)\,dy = c \tag{2}$$

**2. Linear Equations**

A first order equation is called *linear* if it has the form

$$\frac{dy}{dx} + P(x)\,y = Q(x) \tag{3}$$

Multiplying both sides by $e^{\int P\,dx}$, this can be written

$$\frac{d}{dx}\{y\,e^{\int P\,dx}\} = Q\,e^{\int P\,dx}$$

Then integrating, the general solution is

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + c$$

or

$$y = e^{-\int P\,dx}\int Q\,e^{\int P\,dx}\,dx + c\,e^{-\int P\,dx} \tag{4}$$

The factor $e^{\int P\,dx}$ is often called an *integrating factor*.

**3. Exact Equation**

The equation

$$M\,dx + N\,dy = 0 \tag{5}$$

where $M$ and $N$ are functions of $x$ and $y$ is called an *exact differential equation* if $M\,dx + N\,dy$ can be expressed as an exact differential $dU$ of a function $U(x, y)$. In such case the solution is given by $U(x, y) = c$.

A necessary and sufficient condition that (5) be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{6}$$

In some cases an equation is not exact but can be made exact by first multiplying through by a suitably chosen function called an integrating factor as in the case of the linear equation.

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